

# INTEGRABLE MULTIDIMENSIONAL CLASSICAL AND QUANTUM COSMOLOGY FOR INTERSECTING P-BRANES

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Multidimensional cosmological model describing the evolution of one Einstein space of non-zero curvature and  $n$  Ricci-flat internal spaces is considered. The action contains several dilatonic scalar fields  $\varphi^I$  and antisymmetric forms  $A^I$ . When forms are chosen to be proportional of volume forms of  $p$ -brane submanifolds of internal space manifold, the Toda-like Lagrange representation is obtained. Wheeler–De Witt equation for the model is presented. The exact solutions in classical and quantum cases are obtained when dimensions of  $p$ -branes and dilatonic couplings obey some orthogonality conditions.

## 1 Introduction

In this paper we continue our investigations of multidimensional gravitational model governed by the action containing several dilatonic scalar fields and antisymmetric forms [13]. The action is presented below (see (2.1)). Such form of action is typical for special sectors of supergravitational models [1, 2] and may be of interest when dealing with superstring and M-theories [3, 4, 5, 6].

Here we consider a cosmological sector of the model from [13]. We recall that this model treats generalized intersecting  $p$ -brane solutions. Using the  $\sigma$ -model representation of [13] we reduce the equations of motion to the pseudo-Euclidean Toda-like Lagrange system [17] with zero-energy constraint. After separating one-dimensional Liouville subsystem corresponding to negative mode (logarithm of quasivolume [17]) we are led to Euclidean Toda-like system. We consider the simplest case of orthogonal vectors in exponents of the Toda potential and obtain exact solutions. In this case we deal with  $a_1 + \dots + a_1$  Euclidean Toda lattice (sum of  $n$  Liouville actions). Recently analogous reduction for forms of equal rank was done in [8].

In this paper we consider also quantum aspects of the model. Using the  $\sigma$ -model representation and the standard prescriptions of quantization we are led to the multidimensional Wheeler–De Witt equation. This equation is solved in "orthogonal" case.

## 2 The model

Here like in [13] we consider the model governed by the action

$$S = \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} \left\{ R[g] - 2\Lambda - \sum_{I \in \Omega} \left[ g^{MN} \partial_M \varphi^I \partial_N \varphi^I + \frac{1}{n_I!} \exp(2\lambda_{JI} \varphi^J) (F^I)_g^2 \right] \right\} + S_{GH}, \quad (2.1)$$

where  $g = g_{MN} dz^M \otimes dz^N$  is the metric,  $\varphi^I$  is a dilatonic scalar field,

$$F^I = dA^I = \frac{1}{n_I!} F^I_{M_1 \dots M_{n_I}} dz^{M_1} \wedge \dots \wedge dz^{M_{n_I}} \quad (2.2)$$

is  $n_I$ -form ( $n_I \geq 2$ ) on  $D$ -dimensional manifold  $M$ ,  $\Lambda$  is a cosmological constant and  $\lambda_{JI} \in \mathbf{R}$ ,  $I, J \in \Omega$ . In (2.1) we denote  $|g| = |\det(g_{MN})|$ ,

$$(F^I)_g^2 = F^I_{M_1 \dots M_{n_I}} F^I_{N_1 \dots N_{n_I}} g^{M_1 N_1} \dots g^{M_{n_I} N_{n_I}}, \quad (2.3)$$

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and  $S_{\text{GH}}$  is the standard Gibbons-Hawking boundary term [16]. This term is essential for a quantum treatment of the problem. Here  $\Omega$  is a non-empty finite set. (The action (2.1) with  $\Lambda = 0$  and equal  $n_I$  was considered recently in [9]. For supergravity models with different  $n_I$  see also [7]).

The equations of motion corresponding to (2.1) have the following form

$$R_{MN} - \frac{1}{2}g_{MN}R = T_{MN} - \Lambda g_{MN}, \quad T_{MN} = \sum_{I \in \Omega} \left[ T_{MN}^{\varphi^I} + \exp(2\lambda_{JI}\varphi^J)T_{MN}^{F^I} \right], \quad (2.4)$$

$$\Delta[g]\varphi^J - \sum_{I \in \Omega} \frac{\lambda_{JI}}{n_I!} \exp(2\lambda_{KI}\varphi^K) (F^I)_g^2 = 0, \quad \nabla_{M_1}[g] \left( \exp(2\lambda_{KI}\varphi^K) F^{I,M_1 \dots M_{n_I}} \right) = 0, \quad (2.5)$$

where

$$T_{MN}^{\varphi^I} = \partial_M \varphi^I \partial_N \varphi^I - \frac{1}{2}g_{MN} \partial_P \varphi^I \partial^P \varphi^I, \quad (2.6)$$

$$T_{MN}^{F^I} = \frac{1}{n_I!} \left[ -\frac{1}{2}g_{MN} (F^I)_g^2 + n_I F_{MM_2 \dots M_{n_I}}^I F_N^{I,M_2 \dots M_{n_I}} \right], \quad (2.7)$$

$I, J \in \Omega$ . In (2.5)  $\Delta[g]$  and  $\nabla[g]$  are Laplace-Beltrami and covariant derivative operators respectively corresponding to  $g$ .

Here we consider the manifold  $M = \mathbf{R} \times M_0 \times \dots \times M_n$ , with the metric

$$g = w e^{2\gamma(u)} du \otimes du + \sum_{i=0}^n e^{2\phi^i(u)} g^i, \quad (2.8)$$

where  $w = \pm 1$ ,  $u$  is a time variable and  $g^i = g_{m_i n_i}(y_i) dy_i^{m_i} \otimes dy_i^{n_i}$  is a metric on  $M_i$  satisfying the equation  $R_{m_i n_i}[g^i] = \lambda_i g_{m_i n_i}^i$ ,  $m_i, n_i = 1, \dots, d_i$ ;  $\lambda_i = \text{const}$ ,  $i = 0, \dots, n$ . The functions  $\gamma, \phi^i : \mathbf{R}_\bullet \rightarrow \mathbf{R}$  ( $\mathbf{R}_\bullet$  is an open subset of  $\mathbf{R}$ ) are smooth.

We claim any manifold  $M_i$  to be oriented and connected,  $i = 0, \dots, n$ . Then the volume  $d_i$ -form

$$\tau_i = \sqrt{|g^i(y_i)|} dy_i^1 \wedge \dots \wedge dy_i^{d_i}, \quad (2.9)$$

and signature parameter  $\varepsilon(i) = \text{sign}(\det(g_{m_i n_i}^i)) = \pm 1$  are correctly defined for all  $i = 0, \dots, n$ .

Let  $\Omega$  from (2.1) be a set of all non-empty subsets of  $\{0, \dots, n\}$ . The number of elements in  $\Omega$  is  $|\Omega| = 2^{n+1} - 1$ . For any  $I = \{i_1, \dots, i_k\} \in \Omega$ ,  $i_1 < \dots < i_k$ , we put in (2.2)

$$A^I = \Phi^I \tau_{i_1} \wedge \dots \wedge \tau_{i_k}, \quad (2.10)$$

where functions  $\Phi^I : \mathbf{R}_\bullet \rightarrow \mathbf{R}$  are smooth, and  $\tau_i$  are defined in (2.9). In components relation (2.10) reads

$$A_{P_1 \dots P_{d(I)}}^I(u, y) = \Phi^I(u) \sqrt{|g^{i_1}(y_{i_1})|} \dots \sqrt{|g^{i_k}(y_{i_k})|} \varepsilon_{P_1 \dots P_{d(I)}}, \quad (2.11)$$

where  $d(I) \equiv d_{i_1} + \dots + d_{i_k} = \sum_{i \in I} d_i$  is the dimension of the oriented manifold  $M_I = M_{i_1} \times \dots \times M_{i_k}$ , and indices  $P_1, \dots, P_{d(I)}$  correspond to  $M_I$ . It follows from (2.10) that

$$F^I = dA^I = d\Phi^I \wedge \tau_{i_1} \wedge \dots \wedge \tau_{i_k}, \quad (2.12)$$

or, in components,

$$F_{u P_1 \dots P_{d(I)}}^I = -F_{P_1 u \dots P_{d(I)}}^I = \dots = \dot{\Phi}^I \sqrt{|g^{i_1}|} \dots \sqrt{|g^{i_k}|} \varepsilon_{P_1 \dots P_{d(I)}}, \quad (2.13)$$

and  $n_I = d(I) + 1$ ,  $I \in \Omega$ .

Thus dimensions of forms  $F^I$  in the considered model are fixed by the subsequent decomposition of the manifold.

### 3 $\sigma$ -model representation

For dilatonic scalar fields we put  $\varphi^I = \varphi^I(u)$ . Let

$$f = \gamma_0 - \gamma, \quad \sum_{i=0}^n d_i \phi^i \equiv \gamma_0. \quad (3.1)$$

It is not difficult to verify that the field equations (2.4)–(2.5) for the field configurations from (2.8), (2.10) may be obtained as the equations of motion corresponding to the action

$$S_\sigma = \frac{1}{2\kappa_0^2} \int du e^f \left\{ -w G_{ij} \dot{\phi}^i \dot{\phi}^j - w \delta_{IJ} \dot{\varphi}^I \dot{\varphi}^J \right. \\ \left. - w \sum_{I \in \Omega} \varepsilon(I) \exp \left( 2\vec{\lambda}_I \vec{\varphi} - 2 \sum_{i \in I} d_i \phi^i \right) (\dot{\Phi}^I)^2 - 2V e^{-2f} \right\}, \quad (3.2)$$

where  $\vec{\varphi} = (\varphi^I)$ ,  $\vec{\lambda}_I = (\lambda_{JI})$ ,  $\dot{\varphi} \equiv d\varphi(u)/du$ ;  $G_{ij} = d_i \delta_{ij} - d_i d_j$ , are component of "pure cosmological" minisuperspace metric and

$$V = V(\phi) = \Lambda e^{2\gamma_0(\phi)} - \frac{1}{2} \sum_{i=1}^n \lambda_i d_i e^{-2\phi^i + 2\gamma_0(\phi)} \quad (3.3)$$

is the potential. In (3.2)  $\varepsilon(I) \equiv \varepsilon(i_1) \times \dots \times \varepsilon(i_k) = \pm 1$  for  $I = \{i_1, \dots, i_k\} \in \Omega$ .

For finite internal space volumes  $V_i$  (e.g. compact  $M_i$ ) the action (3.2) coincides with the action (2.1) if  $\kappa^2 = \kappa_0^2 \prod_{i=0}^n V_i$ .

The representation (3.2) follows from more general  $\sigma$ -model action from [19], that may be written in the following form

$$S_\sigma = \frac{\mu}{2} \int du \left\{ (-w) \mathcal{N} \mathcal{G}_{\hat{A}\hat{B}}(\sigma) \sigma^{\hat{A}} \sigma^{\hat{B}} - 2\mathcal{N}^{-1} V(\sigma) \right\}, \quad (3.4)$$

where  $(\sigma^{\hat{A}}) = (\phi^i, \varphi^I, \Phi^{I'}) \in \mathbf{R}^{n+1+2m}$ ,  $m = |\Omega|$ ,  $\mu = 1/(2k_0^2)$ ;  $\mathcal{N} = \exp(\gamma_0 - \gamma) > 0$  is the lapse function and

$$(\mathcal{G}_{\hat{A}\hat{B}}) = \begin{pmatrix} G_{ij} & 0 & 0 \\ 0 & \delta_{IJ} & 0 \\ 0 & 0 & \varepsilon(I') \exp \left( 2\vec{\lambda}_{I'} \vec{\varphi} - 2 \sum_{i \in I'} d_i \phi^i \right) \delta_{I'J'} \end{pmatrix} \quad (3.5)$$

is matrix of minisupermetric of the model (target space metric),  $i, j = 0, \dots, n$ ;  $I, J, I', J' \in \Omega$ .

Let us fix the gauge in (3.1):  $f = f(\sigma)$ , where  $f(\sigma)$  is smooth. We call this gauge as  $f$ -gauge. From (3.4) we get the Lagrange system with the Lagrangian and the energy constraint

$$L = \frac{\mu}{2} e^f \mathcal{G}_{\hat{A}\hat{B}}(\sigma) \sigma^{\hat{A}} \sigma^{\hat{B}} + w\mu e^{-f} V(\sigma), \quad (3.6)$$

$$E = \frac{\mu}{2} e^f \mathcal{G}_{\hat{A}\hat{B}}(\sigma) \sigma^{\hat{A}} \sigma^{\hat{B}} - w\mu e^{-f} V(\sigma) = 0. \quad (3.7)$$

We note that the minisupermetric  $\mathcal{G} = \mathcal{G}_{\hat{A}\hat{B}} d\sigma^{\hat{A}} \otimes d\sigma^{\hat{B}}$  is not flat. Here the problem of integrability of Lagrange equations for the Lagrangian (3.6) arises.

The minisuperspace metric (3.5) may be also written in the form

$$\mathcal{G} = \bar{G} + \sum_{I \in \Omega} \varepsilon(I) e^{-2U^I(x)} d\Phi^I \otimes d\Phi^I, \quad U^I(x) = \sum_{i \in I} d_i \phi^i - \vec{\lambda}_I \vec{\varphi}, \quad (3.8)$$

where  $x = (x^A) = (\phi^i, \varphi^I)$ ,  $\bar{G} = \bar{G}_{AB} dx^A \otimes dx^B = G_{ij} d\phi^i \otimes d\phi^j + \delta_{IJ} d\varphi^I \otimes d\varphi^J$ ,

$$(\bar{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & \delta_{IJ} \end{pmatrix} \quad (3.9)$$

$i, j \in \{0, \dots, n\}$ ,  $I, J \in \Omega$ , and the potential  $(-w)V$  may be presented in the following form

$$(-w)V = \sum_{j=0}^n \left( \frac{w}{2} \lambda_i d_i \right) \exp[2U^i(x)] + (-w)\Lambda \exp[2U^\Lambda(x)], \quad (3.10)$$

where

$$U^\Lambda(x) = U_A^\Lambda x^A = \sum_{j=0}^n d_j \phi^j, \quad U^i(x) = -\phi^i + \sum_{j=0}^n d_j \phi^j, \quad (3.11)$$

or in components

$$(U_A^i) = (-\delta_j^i + d_j, 0), \quad (U_A^\Lambda) = (d_j, 0), \quad (U_A^I) = \left( \sum_{i \in I} \delta_j^i d_i, -\lambda_{JI} \right), \quad (3.12)$$

$i, j \in \{0, \dots, n\}$ ,  $I, J \in \Omega$ .

Let  $(x, y) \equiv \bar{G}_{AB} x^A y^B$  define a quadratic form on  $\mathcal{V} = \mathbf{R}^{n+1+m}$ ,  $m = 2^{n+1} - 1$ . The dual form defined on dual space  $\mathcal{V}^*$  is following

$$(U, U')_* = \bar{G}^{AB} U_A U'_B, \quad (3.13)$$

where  $U, U' \in \mathcal{V}^*$ ,  $U(x) = U_A x^A$ ,  $U'(x) = U'_A x^A$  and  $(\bar{G}^{AB})$  is the matrix inverse to the matrix  $(\bar{G}_{AB})$ . Here, like in [15],

$$G^{ij} = \frac{\delta_{ij}}{d_i} + \frac{1}{2-D} \quad (3.14)$$

$i, j = 0, \dots, n$ .

The integrability of the Lagrange system crucially depends upon the scalar products (3.13) for vectors  $U^i$ ,  $U^\Lambda$ ,  $U^I$  from (3.8), (3.11). Here we present these scalar products

$$(U^i, U^j)_* = \frac{\delta_{ij}}{d_j} - 1, \quad (U^i, U^\Lambda)_* = -1, \quad (U^\Lambda, U^\Lambda)_* = -\frac{D-1}{D-2}, \quad (3.15)$$

$$(U^I, U^i)_* = -\frac{d(I \cap \{i\})}{d_i}, \quad (U^I, U^\Lambda)_* = \frac{d(I)}{2-D}, \quad (3.16)$$

$$(U^I, U^J)_* = q(I, J) + \vec{\lambda}_I \vec{\lambda}_J, \quad q(I, J) \equiv d(I \cap J) + \frac{d(I)d(J)}{2-D}, \quad (3.17)$$

$I, J \in \Omega$ ,  $i, j = 0, \dots, n$ .

The relations (3.15) were calculated in [18], the relations (3.17) were obtained in [13] ( $U_A^I = -L_{AI}$  in notations of [13]).

## 4 Classical exact solutions

Here we will integrate the Lagrange equations corresponding to the Lagrangian (3.6) with the energy-constraint (3.7). We put  $f = 0$ , i.e. the harmonic time gauge is considered.

The problem of integrability may be simplified if we integrate the Maxwell equations

$$\frac{d}{du} \left( \exp \left( 2\vec{\lambda}_I \vec{\varphi} - 2 \sum_{i \in I} d_i \phi^i \right) \dot{\Phi}^I \right) = 0, \quad \dot{\Phi}^I = Q^I \exp \left( -2\vec{\lambda}_I \vec{\varphi} + 2 \sum_{i \in I} d_i \phi^i \right), \quad (4.1)$$

where  $Q^I$  are constant,  $I \in \Omega$ .

Let  $Q^I \neq 0 \Leftrightarrow I \in \Omega_*$ , where  $\Omega_* \subset \Omega$  is some non-empty subset of  $\Omega$ . For fixed  $Q = (Q^I, I \in \Omega_*)$  the Lagrange equations corresponding to  $\phi^i$  and  $\varphi^I$ , when equations (4.1) are substituted, are equivalent to the Lagrange equations for the Lagrangian

$$L_Q = \frac{1}{2} \bar{G}_{AB} \dot{x}^A \dot{x}^B - V_Q, \quad (4.2)$$

where  $x = (x^A) = (\phi^i, \varphi^I)$ ,  $i = 0, \dots, n$ ,  $I \in \Omega$  and

$$V_Q = (-w)V + \sum_{I \in \Omega_*} \frac{1}{2} \varepsilon(I) (Q^I)^2 \exp[2U^I(x)], \quad (4.3)$$

(for  $(-w)V$  see (3.10)). Thus, we are led to the pseudo-Euclidean Toda-like system (see [15, 17]) with the zero-energy constraint:

$$E_Q = \frac{1}{2} \bar{G}_{AB} \dot{x}^A \dot{x}^B + V_Q = 0. \quad (4.4)$$

#### 4.1 The case $\Lambda = \lambda_i = 0$ , $i = 1, \dots, n$ .

Here we put  $\Lambda = 0$ ,  $\lambda_i = 0$  for  $i = 1, \dots, n$ ; and  $\lambda_0 \neq 0$ . In this case the potential (4.3)

$$V_Q = \left( \frac{w}{2} \lambda_0 d_0 \right) \exp[2U^0(x)] + \sum_{I \in \Omega_*} \frac{\varepsilon(I)}{2} (Q^I)^2 \exp[2U^I(x)] \quad (4.5)$$

is governed by time-like vector  $U^0$  ( $d_0 > 1$ ) and  $m_* = |\Omega_*|$  space-like vectors  $U^I$ :

$$(U^0, U^0) = \frac{1}{d_0} - 1 < 0, \quad (U^I, U^I) = d(I) \frac{D - 2 - d(I)}{D - 2} + (\vec{\lambda}_I)^2 > 0 \quad (4.6)$$

We also put  $0 \notin I$ ,  $\forall I \in \Omega_*$ . This condition means that all  $p$ -branes do not contain the manifold  $M_0$ . It follows from (3.16) that

$$(U^I, U^0)_* = 0, \quad (4.7)$$

for all  $I \in \Omega_*$ . In this case the Lagrangian (4.2) with the potential (4.5) may be diagonalized by linear coordinate transformation  $z^a = S_i^a x^i$ ,  $a = 0, \dots, n$ , satisfying  $\eta_{ab} S_i^a S_j^b = G_{ij}$ , where  $\eta_{ab} = \text{diag}(-1, +1, \dots, +1)$ . There exists the diagonalization such that  $U_i^0 x^i = q_0 z^0$ , where

$$q_0 = \sqrt{-(U^0, U^0)_*} = \sqrt{1 - \frac{1}{d_0}} \quad (4.8)$$

is a parameter,  $q_0 < 1$ .

In  $z$ -variables the Lagrangian (4.2) reads  $L_Q = L_0 + L_Q^E$ , where

$$L_0 = -\frac{1}{2} (\dot{z}^0)^2 - A_0 \exp(2q_0 z^0), \quad (4.9)$$

$$L_Q^E = \frac{1}{2} \left[ (\dot{\vec{z}})^2 + \delta_{IJ} \dot{\varphi}^I \dot{\varphi}^J \right] - \sum_{I \in \Omega_*} A_I \exp \left( 2\vec{q}_I \vec{z} - 2\lambda_{JI} \varphi^J \right). \quad (4.10)$$

In (4.9), (4.10)  $\vec{z} = (z^1, \dots, z^n)$ ,  $\vec{q}_I = (q_{I,1}, \dots, q_{I,n})$ ,  $A_0 \equiv (w/2)\lambda_0 d_0$ ,  $A_I = (1/2)\varepsilon(I)(Q^I)^2$  and  $\vec{q}_I \cdot \vec{q}_J = q(I, J)$ ,  $I, J \in \Omega_*$ .

Thus the Lagrangian (4.2) is splitted into sum of two independent parts (4.9) and (4.10). The latter may be written as

$$L_Q^E = \frac{1}{2}(\dot{\vec{Z}})^2 - \sum_{I \in \Omega_*} A_I \exp(2\vec{B}_I \vec{Z}) \quad (4.11)$$

where  $\vec{Z} = (z^1, \dots, z^n, \varphi^I)$ ,  $\vec{B}_I = (\vec{q}_I, \lambda_{JI})$ . Thus the equations of motion for the considered cosmological model are reduced to the equations of motion for the Lagrange systems with the Lagrangians (4.9) and (4.10) and the energy constraint  $E = E_0 + E_Q^E = 0$ , where

$$E_0 = -\frac{1}{2}(\dot{z}^0)^2 + A_0 \exp(2q_0 z^0), \quad E_Q^E = \frac{1}{2}(\dot{\vec{Z}})^2 + \sum_{I \in \Omega_*} A_I \exp(2\vec{B}_I \vec{Z}). \quad (4.12)$$

The vectors  $\vec{B}_I$  in (4.11) satisfy the relations

$$\vec{B}_I \cdot \vec{B}_J = q(I, J) + \vec{\lambda}_I \vec{\lambda}_J, \quad (4.13)$$

$I, J \in \Omega_*$ , where  $p$ -brane "overlapping index"  $q(I, J)$  is defined in (3.17).

## 4.2 The case of orthogonal $\vec{B}_I$

The simplest situation arises when the vectors  $\vec{B}_I$  are orthogonal, i.e.

$$\vec{B}_I \vec{B}_J = (U^I, U^J)_* = d(I \cap J) + \frac{d(I)d(J)}{2 - D} + \vec{\lambda}_I \vec{\lambda}_J = 0, \quad (4.14)$$

for all  $I \neq J$ ,  $I, J \in \Omega_*$ . In this case the Lagrangian (4.11) may be splitted into the sum of  $|\Omega_*|$  Lagrangians of the Liouville type and  $n$  "free" Lagrangians.

Using relations from [18] we readily obtain exact solutions for the Euler-Lagrange equations corresponding to the Lagrangian (4.2) with the potential (4.5) when the orthogonality conditions (4.7) and (4.14) are satisfied.

The solutions for  $(x^A) = (\phi^i, \varphi^I)$  read

$$x^A(u) = -\frac{U^{0A}}{(U^0, U^0)_*} \ln |f_0(u - u_0)| - \sum_{I \in \Omega_*} \frac{U^{IA}}{(U^I, U^I)_*} \ln |f_I(u - u_I)| + \alpha^A u + \beta^A, \quad (4.15)$$

where  $u_0, u_I$  are constants,  $U^{sA} \equiv \bar{G}^{AB} U_B^s$  are contravariant components of  $U^s$ ,  $s \in \{0\} \sqcup \Omega_*$ ,  $\bar{G}^{AB}$  is the matrix inverse to the matrix (3.9). Functions  $f_0$ ,  $f_I$  in (4.15) are the following

$$f_0(\tau) = \begin{cases} \left| \frac{\lambda_0(d_0 - 1)}{C_0} \right|^{\frac{1}{2}} \operatorname{sh}(\sqrt{C_0}\tau), & C_0 > 0, \lambda_0 w > 0; \\ \left| \frac{\lambda_0(d_0 - 1)}{C_0} \right|^{\frac{1}{2}} \operatorname{ch}(\sqrt{C_0}\tau), & C_0 > 0, \lambda_0 w < 0; \end{cases} \quad \begin{cases} \left| \frac{\lambda_0(d_0 - 1)}{C_0} \right|^{\frac{1}{2}} \sin(\sqrt{|C_0|}\tau), & C_0 < 0, \lambda_0 w > 0; \\ |\lambda_0(d_0 - 1)|^{\frac{1}{2}} \tau, & C_0 = 0, \lambda_0 w > 0; \end{cases} \quad (4.16)$$

and

$$f_I(\tau) = \begin{cases} \frac{|Q^I|}{\nu_I |C_I|^{1/2}} \operatorname{sh}(\sqrt{C_I}\tau), & C_I > 0, \varepsilon(I) < 0; \\ \frac{|Q^I|}{\nu_I |C_I|^{1/2}} \operatorname{ch}(\sqrt{C_I}\tau), & C_I > 0, \varepsilon(I) > 0; \end{cases} \quad \begin{cases} \frac{|Q^I|}{\nu_I |C_I|^{1/2}} \sin(\sqrt{|C_I|}\tau), & C_I < 0, \varepsilon(I) < 0; \\ \frac{|Q^I|}{\nu_I} \tau, & C_I = 0, \varepsilon(I) < 0, \end{cases} \quad (4.17)$$

where  $C_0, C_I$  are constants, and

$$\nu_I^{-1} = \sqrt{d(I) \left( 1 + \frac{d(I)}{2-D} \right) + \vec{\lambda}_I^2} > 0, \quad (4.18)$$

$I \in \Omega_*$ .

Vectors  $\alpha = (\alpha^A)$  and  $\beta = (\beta^A)$  in (4.15) satisfy the linear constraint relations:

$$U^0(\alpha) = -\alpha^0 + \sum_{j=0}^n d_j \alpha^j = 0; \quad U^0(\beta) = -\beta^0 + \sum_{j=0}^n d_j \beta^j = 0; \quad (4.19)$$

$$U^I(\alpha) = \sum_{i \in I} d_i \alpha^i - \lambda_{JI} \alpha^J = 0; \quad U^I(\beta) = \sum_{i \in I} d_i \beta^i - \lambda_{JI} \beta^J = 0. \quad (4.20)$$

Calculations of contravariant components  $U^{sA}$  give the following relations

$$U^{0i} = -\frac{\delta_0^i}{d_i}, \quad U^{0I} = 0, \quad U^{Ii} = \sum_{k \in I} \delta^{ik} + \frac{d(I)}{2-D}, \quad U^{IJ} = -\lambda_{JI}, \quad (4.21)$$

$i = 0, \dots, n$ ;  $J, I \in \Omega_*$ . Substitution of (4.21) and (4.6) into the solution (4.15) leads us to the following relations for the logarithms of scale fields

$$\phi^i = \frac{\delta_0^i}{1-d_0} \ln |f_0| + \sum_{I \in \Omega_*} \alpha_I^i \ln |f_I| + \alpha^i u + \beta^i, \quad \varphi^J = \sum_{I \in \Omega_*} \lambda_{JI} \nu_I^2 \ln |f_I| + \alpha^J u + \beta^J, \quad (4.22)$$

where  $\alpha_I^i = -\left(\sum_{k \in I} \delta^{ik} + d(I)/(2-D)\right) \nu_I^2$ ,  $i = 0, \dots, n$ ;  $I \in \Omega_*$ .

For harmonic gauge  $\gamma = \gamma_0(\phi)$  we get from (4.22)

$$\gamma = \sum_{i=0}^n d_i \phi^i = \frac{d_0}{1-d_0} \ln |f_0| + \sum_{I \in \Omega_*} \frac{d(I)}{D-2} \nu_I^2 \ln |f_I| + \alpha^0 u + \beta^0, \quad (4.23)$$

where  $\alpha^0$  and  $\beta^0$  are given by (4.19).

The zero-energy constraint

$$E = E_0 + \sum_{I \in \Omega_*} E_I + \frac{1}{2}(\alpha, \alpha) = 0, \quad (4.24)$$

where  $\alpha = (\alpha^i, \alpha^I)$ ,  $(\alpha, \alpha) = \bar{G}_{AB} \alpha^A \alpha^B = G_{ij} \alpha^i \alpha^j + \delta_{IJ} \alpha^I \alpha^J$  and  $C_s = 2E_s(U^s, U^s)_*$ ,  $s = 0, I$ , (see [18]) may be written in the following form

$$C_0 \frac{d_0}{d_0-1} = \sum_{I \in \Omega_*} [C_I \nu_I^2 + (\alpha^I)^2] + \sum_{i=1}^n d_i (\alpha^i)^2 + \frac{1}{d_0-1} \left( \sum_{i=1}^n d_i \alpha^i \right)^2. \quad (4.25)$$

The substitution of relations (4.22) into (4.1) implies  $\dot{\Phi}^I = Q_I/f_I^2$  and hence we get for forms

$$F^I = d\Phi^I \wedge d\tau_I = \frac{Q^I}{f_I^2} du \wedge d\tau_I, \quad (4.26)$$

where  $\tau_I \equiv \tau_{i_1} \wedge \dots \wedge \tau_{i_k}$ ,  $I = \{i_1, \dots, i_k\} \in \Omega_*$ ,  $i_1 < \dots < i_k$ .

The relation for the metric may be readily obtained using the formulas (4.22), (4.23).

$$\begin{aligned} g = & \left( \prod_{I \in \Omega_*} [f_I^2(u-u_I)]^{d(I)\nu_I^2/(D-2)} \right) \left\{ [f_0^2(u-u_0)]^{d_0/(1-d_0)} e^{2\alpha^0 u + 2\beta^0} \right. \\ & \times [wdu \otimes du + f_0^2(u-u_0)g^0] + \sum_{i \neq 0} \left( \prod_{I \in \Omega_* \setminus \{i\}} [f_I^2(u-u_I)]^{-\nu_I^2} \right) e^{2\alpha^i u + 2\beta^i} g^i \left. \right\}. \end{aligned} \quad (4.27)$$

## 5 Wheeler–De Witt equation

Let us consider the Lagrangian system with Lagrangian (3.6). Using the standard prescriptions of quantization (see, for example, [15]) we are led to the Wheeler–DeWitt equation

$$\hat{H}^f \Psi^f \equiv \left( -\frac{1}{2\mu} \Delta [\mathcal{G}] + \frac{a}{\mu} R[\mathcal{G}] + e^{-f} \mu(-w) V \right) \Psi^f = 0, \quad (5.1)$$

where

$$a = \frac{N-2}{8(N-1)}, \quad N = n+1+2|\Omega|. \quad (5.2)$$

Here  $\Psi^f = \Psi^f(\sigma)$  is the so-called "wave function of the universe" corresponding to the  $f$ -gauge,  $\Delta[\mathcal{G}]$  and  $R[\mathcal{G}]$  denote the Laplace–Beltrami operator and the scalar curvature corresponding to  $\mathcal{G}_1$ . For the scalar curvature we get

$$R[\mathcal{G}] = - \sum_{I \in \Omega} (U^I, U^I) - \sum_{I, I' \in \Omega} (U^I, U^{I'}). \quad (5.3)$$

For the Laplace operator we obtain

$$\Delta[\mathcal{G}] \equiv \frac{1}{\sqrt{|\mathcal{G}|}} \partial_{\hat{A}} \left( \mathcal{G}^{\hat{A}\hat{B}} \sqrt{|\mathcal{G}|} \partial_{\hat{B}} \right) = e^{U(x)} \frac{\partial}{\partial x^A} \left( \bar{G}^{AB} e^{-U(x)} \frac{\partial}{\partial x^B} \right) + \sum_{I \in \Omega} \varepsilon(I) e^{2U^I(x)} \left( \frac{\partial}{\partial \Phi^I} \right)^2, \quad (5.4)$$

where

$$U(x) = \sum_{I \in \Omega} U^I(x). \quad (5.5)$$

**Harmonic-time gauge.** The WDW equation (5.1) for  $f = 0$

$$\hat{H} \Psi \equiv \left( -\frac{1}{2\mu} \Delta[\mathcal{G}] + \frac{a}{\mu} R[\mathcal{G}] + \mu(-w) V \right) \Psi = 0, \quad (5.6)$$

may be rewritten, using relations (5.3), (5.4) as follows

$$\begin{aligned} 2\mu \hat{H} \Psi &= \left\{ -G^{ij} \frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi^j} - \delta^{IJ} \frac{\partial}{\partial \varphi^I} \frac{\partial}{\partial \varphi^J} - \sum_{I \in \Omega} \varepsilon(I) e^{2U^I(x)} \left( \frac{\partial}{\partial \Phi^I} \right)^2 \right. \\ &\quad \left. + \sum_{I \in \Omega} \left[ \sum_{i \in I} \frac{\partial}{\partial \phi^i} - \frac{d(I)}{D-2} \sum_{j=0}^n \frac{\partial}{\partial \phi^j} - \lambda_{JI} \frac{\partial}{\partial \varphi^J} \right] + 2aR[\mathcal{G}] + 2\mu^2(-w)V \right\} \Psi = 0. \end{aligned} \quad (5.7)$$

Here  $\hat{H} \equiv \hat{H}^{f=0}$  and  $\Psi = \Psi^{f=0}$ .

## 6 Quantum solutions

Let us now consider the solutions of the Wheeler–De Witt equation (5.6) for the case of harmonic-time gauge. We also note that the orthogonality conditions (4.7) and (4.14) are satisfied. In  $z$ -variables  $z = (z^A) = (S_B^A x^B) = (z^0, \vec{z}, z^I)$  satisfying  $q_0 z^0 = U^0(x)$ ,  $q_I z^I = U^I(x)$ ,  $q_I = \nu_I^{-1}$ , we get

$$\Delta[\mathcal{G}] = - \left( \frac{\partial}{\partial z^0} \right)^2 + \left( \frac{\partial}{\partial \vec{z}} \right)^2 + \sum_{I \in \Omega} e^{q_I z^I} \frac{\partial}{\partial z^I} \left( e^{-q_I z^I} \frac{\partial}{\partial z^I} \right) + \sum_{I \in \Omega} \varepsilon(I) e^{2q_I z^I} \left( \frac{\partial}{\partial \Phi^I} \right)^2. \quad (6.1)$$

The relation (5.3) in the orthogonal case reads as

$$R[\mathcal{G}] = -2 \sum_{I \in \Omega} (U^I, U^I) = -2 \sum_{I \in \Omega} q_I^2. \quad (6.2)$$

We are seeking the solution to WDW equation (5.6) by the method of separation of variables, i.e. we put

$$\Psi_*(z) = \Psi_0(z^0) \left( \prod_{I \in \Omega} \Psi_I(z^I) \right) e^{i P_I \Phi^I} e^{i \vec{p} \vec{z}}. \quad (6.3)$$

It follows from (6.1) that  $\Psi_*(z)$  satisfies the WDW equation (5.6) if

$$2\hat{H}_0 \Psi_0 \equiv \left\{ \left( \frac{\partial}{\partial z^0} \right)^2 + \mu^2 w \lambda_0 d_0 e^{2q_0 z^0} \right\} \Psi_0 = 2\mathcal{E}_0 \Psi_0; \quad (6.4)$$

$$2\hat{H}_I \Psi_I \equiv \left\{ -e^{q_I z^I} \frac{\partial}{\partial z^I} \left( e^{-q_I z^I} \frac{\partial}{\partial z^I} \right) + \varepsilon(I) P_I^2 e^{2q_I z^I} \right\} \Psi_I = 2\mathcal{E}_I \Psi_I, \quad (6.5)$$

$I \in \Omega$ , and

$$2\mathcal{E}_0 + (\vec{p})^2 + 2 \sum_{I \in \Omega} \mathcal{E}_I + 2aR[\mathcal{G}] = 0, \quad (6.6)$$

with  $a$  and  $R[\mathcal{G}]$  from (5.2) and (6.2) respectively.

Solving (6.4), (6.5) we obtain

$$\Psi_0(z^0) = B_{\omega_0(\mathcal{E}_0)}^0 \left( \mu \sqrt{-w \lambda_0 d_0} \frac{e^{q_0 z^0}}{q_0} \right), \quad \Psi_I(z^I) = e^{q_I z^I/2} B_{\omega_I(\mathcal{E}_I)}^I \left( \sqrt{-\varepsilon(I) P_I^2} \frac{e^{q_I z^I}}{q_I} \right), \quad (6.7)$$

where  $\omega_0(\mathcal{E}_0) = \sqrt{2\mathcal{E}_0}/q_0$ ,  $\omega_I(\mathcal{E}_I) = \sqrt{1/4 - \varepsilon(I) 2\mathcal{E}_I \nu_I^2}$ ,  $I \in \Omega$  ( $\nu_I = q_I^{-1}$ ) and  $B_\omega^0, B_\omega^I = I_\omega, K_\omega$  are modified Bessel function.

The general solution of the WDW equation (5.6) is a superposition of the "separated" solutions (6.3):

$$\Psi(z) = \sum_B \int dp dP d\mathcal{E} C(P, p, \mathcal{E}, B) \Psi_*(z|P, p, \mathcal{E}, B), \quad (6.8)$$

where  $p = (\vec{p})$ ,  $P = (P_I)$ ,  $\mathcal{E} = (\mathcal{E}_I)$ ,  $B = (B^0, B^I)$ ,  $B^0, B^I = I, K$ ,  $\Psi_* = \Psi_*(z|P, p, \mathcal{E}, B)$  are given by relation (6.3), (6.7) with  $\mathcal{E}_0$  from (6.6).

## 7 Conclusion

Thus we obtained exact classical and quantum solutions for multidimensional cosmology, describing the evolution of  $(n+1)$  spaces  $(M_0, g_0), \dots, (M_n, g_n)$ , where  $(M_0, g_0)$  is an Einstein space of non-zero curvature, and  $(M_i, g^i)$  are "internal" Ricci-flat spaces,  $i = 1, \dots, n$ ; in the presence of several scalar fields and forms.

The classical solution is given by relations (4.22), (4.26), (4.27) with the functions  $f_0$ ,  $f_I$  defined in (4.16)–(4.17) and the relations on the parameters of solutions  $\alpha^s$ ,  $\beta^s$ ,  $C_s$  ( $s = i, I$ ),  $\nu_I$ , (4.19)–(4.20), (4.18), (4.25) imposed. The quantum solutions are presented by relations (6.3), (6.6)–(6.8).

These solutions describe a set of charged (by forms) overlapping  $p$ -branes "living" on submanifolds not containing internal space  $M_0$ . The solutions are valid if the dimensions of  $p$ -branes and dilatonic coupling vector satisfy the orthogonality restrictions (4.14).

The special case  $w = +1$ ,  $M_0 = S^{d_0}$  corresponds to spherically-symmetric configurations containing black hole solutions (see, for example [10, 11, 12]). It may be interesting to apply the relations from Sect. 6 to minisuperspace quantization of black hole configurations in string models, M-theory etc.

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